Prop There is a symmetric monoidal equivalence $QCoh(B(S^1_{fil})^{gr}) \simeq Mod_\Lambda$.

Recall $\Lambda := k[\varepsilon]$ graded dg Hopf algebra $\left(\begin{array}{l} \varepsilon^2 = 0, \\ \Delta(\varepsilon) = \varepsilon \otimes 1 - 1 \otimes \varepsilon, \\ \varepsilon \text{ degree } -1, \text{ weight } 1 \end{array} \right)$

Mod_Λ^{str} = symmetric monoidal 1-category of Λ-modules in graded cochain complexes

$Mod_\Lambda = Mod_\Lambda^{str}[q\text{-iso}^{-1}]$ = symmetric monoidal ∞-category of Λ-modules in graded derived category

<u>Rmk</u> (a) really want an equiv of graded ∞-categories

$$\begin{tikzcd}
\mathrm{Mod}_\Lambda \arrow[rr, dashed, "\Psi"] & & \mathrm{QCoh}(B(S^1_{\mathrm{fil}})^{gr}) = \mathcal{C} \\
& \mathrm{Mod}_k^{gr} \simeq \mathrm{QCoh}(B\mathbb{G}_m) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

restriction along counit $\rightsquigarrow$

$\Lambda \to k$

(b) graded ∞-category $\mathcal{C}$ $\rightsquigarrow$ graded modules of maps

$$\underline{\mathrm{Map}}_{\mathcal{C}}(X,Y)^n \simeq \mathrm{Map}_{\mathcal{C}}(X, Y(n))$$

(c) Given ψ, how do we check it's an equivalence?

$\Longleftarrow$ $X = \psi(\Lambda)$ is a "graded compact generator"

(i) $\underline{\mathrm{Map}}_{\mathcal{C}}(X, -)$ preserve colimits

(ii) $\underline{\mathrm{Map}}_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow \mathrm{QCoh}(B\mathbb{G}_m)$ conservative

Proof of prop (1) $(S^1_{fil})^{gr} \simeq [BKer/\mathbb{G}_m]$ $\leftarrow$

$\rightsquigarrow$ want to construct $\psi : \mathrm{Mod}_\Lambda \longrightarrow \mathrm{QCoh}([B^2Ker/\mathbb{G}_m])$

(2) Λ is dualizable with

$\Lambda^\vee \simeq k[\eta] \simeq H^*(BKer; G)$ as graded $\mathbb{G}$ Hopf algebra

$\simeq R\Gamma(BKer; G)$ in derived category

G finite group

$\mathrm{QCoh}(BG) \simeq \mathrm{Comod}_{k^G} \simeq \mathrm{Mod}_{k[G]} \simeq \mathrm{Rep}(G)$

(3) $$\mathrm{Mod}^{str}_{\Lambda} \underset{\text{"id on object"}}{\simeq} \mathrm{Comod}^{str}_{\Lambda^\vee} \simeq \lim_{\Delta} \mathrm{Mod}^{str}_{(\Lambda^\vee)^{\otimes n}}$$

$$\Lambda \otimes M \to M$$

$$M \to \Lambda^\vee \otimes M$$

$$M \longmapsto \left(M \otimes (\Lambda^\vee)^{\otimes n}\right)$$

$$\mathrm{Mod}_\Lambda = (\mathrm{Mod}^{str}_\Lambda)[q\text{iso}^{-1}] \simeq \left(\lim_\Delta \mathrm{Mod}^{str}_{(\Lambda^\vee)^{\otimes n}}\right)[q\text{iso}^{-1}]$$

$$\xrightarrow{\sim} \lim_\Delta \left(\mathrm{Mod}^{str}_{(\Lambda^\vee)^{\otimes n}}[q\text{iso}^{-1}]\right)$$

$$\simeq \lim_\Delta \mathrm{Mod}^{gr}_{R\Gamma(B\mathrm{Ker}_i, \mathcal{O})^{\otimes n}}$$

$$\xrightarrow{\sim} \lim_\Delta \mathrm{QCoh}\left([B\mathrm{Ker}^n / \mathbb{G}_m]\right) \simeq \mathrm{QCoh}\left([B^2\mathrm{Ker}/\mathbb{G}_m]\right)$$

without gr^{-1}.
proved §3
graded statement follows $\rightsquigarrow$

(4) Check that the composite ψ sends Λ to a graded cpct gen

$$\underline{\mathrm{Map}}_{\lim_{\Delta} \mathrm{Mod}^{gr}_{(\Lambda^\vee)^{\otimes n}}}\left(\psi'(\Lambda), Y\right)$$

$Y_\bullet = (Y_n)$

$Y_n = (\Lambda^\vee)^{\otimes n} \otimes Y_0$

$$\simeq \lim_{\Delta} \underline{\mathrm{Map}}_{\mathrm{Mod}^{gr}_{\underline{(\Lambda^\vee)^{\otimes n}}}}\left((\underline{\Lambda^\vee})^{\otimes n} \otimes \Lambda, (\Lambda^\vee)^{\otimes n} \otimes Y_0\right)$$

$$\simeq \lim_{\Delta} \underline{\mathrm{Map}}_{\mathrm{Mod}^{gr}_k}\left(\Lambda, (\Lambda^{\vee})^{\otimes n} \otimes Y_0\right)$$

$$\simeq \lim_{\Delta} (\Lambda^\vee)^{\otimes n+1} \otimes Y_0 \simeq Y_0 \in \mathrm{Mod}^{gr}_k$$

We see the zenith projection functor

$$\lim_{\Delta} \mathrm{Mod}^{gr}_{(\Lambda^\vee)^{\otimes n}} \longrightarrow \mathrm{Mod}^{gr}_k$$

which is conservative and preserves colimits

In the paper to finish they cite the result

$$\mathrm{End}_{\mathrm{QCoh}(BS^1_{fil})}(\mathcal{O}) \simeq R\Gamma(BS^1_{fil};\mathcal{O}) \simeq k[u]$$